Deriving Demand Functions - Examples\footnote{Disclaimer: This handout has not been reviewed by the professor. In the case of any discrepancy between this handout and lecture material, the lecture material should be considered the correct source. Despite all efforts, typos may find their way in - please read with a wary eye.}

What follows are some examples of different preference relations and their respective demand functions. In all the following examples, assume we have two goods $x_1$ and $x_2$, with respective prices $p_1$ and $p_2$, and income $m$.

1 Perfect Substitutes

For perfect substitutes, we have to look at respective prices. After all, if goods are perfect substitutes, then the consumer is indifferent between them, and will have no problem adjusting consumption to get the good with the lowest price.

1.1 The basic case (1:1)

For 1:1 perfect substitutes, the situation is about as plain as can be. Say $p_1 > p_2$. The consumer will spend all their income on good 2. How do we know without doing any of that fancy math stuff? If the consumer is just as happy with a unit of good 1 as they are with a unit of good 2, and good 2 is less expensive, then they might as well use all their income on good 2 (they get more stuff that way). Similarly, if $p_1 < p_2$, the consumer will choose only good 1. What if $p_1 = p_2$? Then any combination of good 1 and good 2 that uses all their budget is fine with them. So for each good, we have three possible demand functions depending on the prices. For example, demand for good 1 can be expressed as

$$x_1(p_1,p_2,m) = \begin{cases} 0 & \text{if } p_1 > p_2 \\ \frac{m}{p_1} & \text{if } p_1 < p_2 \\ \text{Any } (x_1,x_2) \text{ that satisfies } p_1 x_1 + p_2 x_2 = m & \text{if } p_1 = p_2 \end{cases}$$

and similarly for good 2 (with the inequalities reversed, of course).

1.2 A more complicated example (2:3)

\textbf{Problem:} Let the individual have a utility function $u(x_1, x_2) = 2x_1 + 3x_2$ and an income of 120. They face prices $p_1 = 2$ and $p_2 = 6$. What is their demand for $x_1$? For $x_2$?
Solution: The easiest way to do this is to look at how much \( x_1 \) they can buy with all their income and how much \( x_2 \) they can buy with all their income, then see which gives the higher utility. If they spend all their money on \( x_1 \), their utility is 
\[
    u(x_1, 0) = 2 \cdot \frac{m}{p_1} = 2 \cdot \frac{120}{2} = 120.
\]
If they spend all their income on \( x_2 \), their utility is 
\[
    u(0, x_2) = 3 \cdot \frac{m}{p_2} = 3 \cdot \frac{120}{6} = 60.
\]
Since they get a higher utility from consuming only \( x_1 \), their demand functions will be 
\[
    x_1(p_1, p_2, m) = \frac{m}{p_1} \quad \text{and} \quad x_2(p_1, p_2, m) = 0.
\]

We could also have solved this by figuring out the slope of the budget line relative to the slope of the indifference curves (i.e. the MRS). The slope of the budget line is 
\[
    -\frac{p_1}{p_2} = -\frac{1}{3},
\]
and the MRS is \(-2/3\). Since the absolute value of the price ratio is lower than the absolute value of the MRS at all point, we know the individual is never going to trade \( x_1 \) for \( x_2 \), so they’ll spend all their income on \( x_1 \) - the same result we found above.

At what price would the individual would be willing to consume a non-zero amount of \( x_2 \)? Using our first method, that would happen whenever 
\[
    u(x_1, 0) = 120 \leq 3 \cdot \frac{120}{p_2} \quad \text{or} \quad p_2 \leq 3.
\]
Using our second method, we know they will be willing to trade some \( x_1 \) for some \( x_2 \) when the absolute value of the price ratio is at least as great as the absolute value of the MRS, or when 
\[
    \frac{2}{p_2} \geq \frac{2}{3},
\]
again giving 
\[
    p_2 \leq 3.
\]

2 Perfect Compliments

With perfect compliments, we solve for the demand functions using two different equations. The first equation is the budget line, and the second is the optimal relationship between the amounts of \( x_1 \) and \( x_2 \) consumed.

2.1 The basic case (1:1)

When goods are perfect 1:1 compliments, the optimal relationship between \( x_1 \) and \( x_2 \) is such that \( x_1 = x_2 \). Why? Say \( x_1 > x_2 \). Our utility is going to be the minimum of the two values \( (x_2) \) and any \( x_1 \) above \( x_2 \) isn’t giving us any additional utility, even though it is eating up some of our budget. We could increase our utility by getting rid of some of that extra \( x_1 \) and buying more \( x_2 \). We’d be in a similar situation if \( x_1 < x_2 \). The only time we couldn’t modify our consumption to improve our utility would be when \( x_1 = x_2 \), so that must be the optimal relation. In general, if you have a utility function of the form 
\[
    \min \{ \alpha, \beta \},
\]

a condition of optimality is that \( \alpha = \beta \).
When we’re optimizing we’ll spend all our income, so we also know that $p_1 x_1 + p_2 x_2 = m$. With these two equations, we can solve for one good in terms of the other, then use the budget equation to solve for the demand functions, like so:

\[ x_1 = x_2 \quad \text{(optimal relation condition)} \]  \hspace{1cm} (1)

\[ p_1 x_1 + p_2 x_2 = m \quad \text{(budget set)} \]  \hspace{1cm} (2)

Plugging (1) into (2), we get

\[ p_1 x_2 + p_2 x_2 = m \quad \Rightarrow \quad x_1(p_1, p_2, m) = \frac{m}{p_1 + p_2} \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{m}{p_1 + p_2} \]

where we get the demand for $x_1$ by again using (1).

2.2 Some more complicated examples

\textbf{Problem} : The individual has a utility function $u(x_1, x_2) = \min\{4x_1, 5x_2\}$ and faces prices $p_1 = 10$ and $p_2 = 5$. We know they consume 20 units of $x_2$ and spend all their income. What is the demand for $x_1$? What is the individual’s income?

\textbf{Solution} : We know that at the optimal point, the individual will choose $x_1$ and $x_2$ in a ratio such that $4x_1 = 5x_2$. Thus

\[ x_1 = \frac{5}{4} x_2 \]

\[ = \frac{5}{4} \times 20 = 25 \]

We know that at the optimal point $p_1 x_1 + p_2 x_2 = m$, and we know everything on the left hand side, so

\[ 10 \times 25 + 5 \times 20 = m = 350 \]

\textbf{Problem} : The individual has a utility function $u(x_1, x_2) = \min\{x_1 + 3x_2, x_2 + 4x_1\}$, faces prices $p_1 = 3$ and $p_2 = 8$, and has an income of 300. Find their demand for $x_1$ and $x_2$.

\textbf{Solution} : Using the tricks discussed above, we get our two equations

\[ x_1 + 3x_2 = x_2 + 4x_1 \quad \Rightarrow \quad x_1 = \frac{2}{3} x_2 \]  \hspace{1cm} (3)

\[ p_1 x_1 + p_2 x_2 = m \]  \hspace{1cm} (4)
Plugging (3) into (4) and solving

$$m = \frac{p_1}{3} + p_2 x_2 \Rightarrow x_1(p_1, p_2, m) = \frac{2m}{2p_1 + 3p_2} \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{3m}{2p_1 + 3p_2}$$

where we got $x_1$ by plugging the optimal $x_2$ into (3). Replacing $p_1$, $p_2$, and $m$ with their known values, we get our quantities demanded.

$$x_1(p_1, p_2, m) = \frac{600}{2 \times 3 + 3 \times 8} = 20 \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{900}{2 \times 3 + 3 \times 8} = 30$$

### 2.3 A general result

In general, if a utility function is of the form

$$\min\{\alpha x_1, \beta x_2\}$$

then the demand functions will be

$$x_1(p_1, p_2, m) = \frac{\beta m}{\beta p_1 + \alpha p_2} \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{\alpha m}{\beta p_1 + \alpha p_2}$$

### 3 Cobb-Douglas

You should be plenty comfortable with Cobb-Douglas preferences by now - we’ve dealt with them quite a bit. Since it’s familiar territory, this first example will be speedy, and then we’ll solve the same problem using the Lagrange method.

**Problem** : Let someone have a utility function defined by $u(x_1, x_2) = \frac{3}{8} x_1^5 x_2$. They have an income of 150 and face prices $p_1 = 10$ and $p_2 = 5$. What is their demand for both goods? What is the ratio of total expenditure on good 1 to expenditure on good 2?

**Solution** : First solve for the optimality condition by setting the negative of the price ratio equal to the marginal rate of substitution. We then use that to get $x_1$ in terms of $x_2$ (or the other way around, whichever suits you).

$$-\frac{p_1}{p_2} = -\frac{5x_1^4 x_2}{x_1^5}$$

$$1 = 5 \frac{x_2 p_2}{x_1 p_1}$$

$$x_1 = 5x_2 \frac{p_2}{p_1}$$
Now plug this into the budget constraint and solve.

\[ m = 5p_1x_2 \frac{p_2}{p_1} + p_2x_2 \]

Now we have our demand functions

\[ x_1(p_1, p_2, m) = \frac{5m}{6p_1} = \frac{5 \times 150}{6 \times 10} = 12.5 \]
\[ x_2(p_1, p_2, m) = \frac{m}{6p_2} = \frac{150}{6 \times 5} = 5 \]

The ratio of expenditure on good 1 to expenditure on good 2 is

\[ \frac{p_1x_1}{p_2x_2} = \frac{10 \times 12.5}{5 \times 5} = 5 \]

which means that however much money we spend on good 1, we spend 5 times that amount on good 2. Remember back when we said that with a utility function representing Cobb-Douglas preferences of the form \( u(x_1, x_2) = cx_1^\alpha x_2^{1-\alpha} \) where \( 0 < \alpha < 1 \), the \( \alpha \) and \( (1-\alpha) \) told you what fraction of your income you spent on each good? We just verified that result. If we transformed our utility function by raising it to the power of \( \frac{1}{6} \) (remember why?) our exponents would be \( \frac{5}{6} \) on good 1 and \( \frac{1}{6} \) on good 2, which would mean that we spend 5 times as much on good 1 as good 2 . . . ta-da!

### 3.1 Solving the problem using the Lagrange method

Following the method shown in class (from the lecture on 11/1), we’re going to solve

\[ \text{maximize } u(x_1, x_2) = \frac{3}{8}x_1^5 x_2 \text{ subject to } p_1x_1 + p_2x_2 = m \]

Forming the Lagrangian gives

\[ L(x_1, x_2, \lambda) = \frac{3}{8}x_1^5 x_2 - \lambda(p_1x_1 + p_2x_2 - m) \]

We now take first derivatives and solve for first order conditions in \( x_1, x_2, \) and \( \lambda \).

\[ \frac{\partial L}{\partial x_1} = \frac{3}{8}5x_1^4 x_2 - \lambda p_1 = 0 \quad \Rightarrow \quad \frac{3}{8}5x_1^4 x_2 = \lambda p_1 \quad (6) \]
\[ \frac{\partial L}{\partial x_2} = \frac{3}{8}x_1^5 - \lambda p_2 = 0 \quad \Rightarrow \quad \frac{3}{8}x_1^5 = \lambda p_2 \quad (7) \]
\[ \frac{\partial L}{\partial \lambda} = p_1x_1 + p_2x_2 - m = 0 \quad \Rightarrow \quad p_1x_1 + p_2x_2 = m \quad (8) \]
Dividing (6) by (7) gives
\[
\frac{p_1}{p_2} = 5 \frac{x_2}{x_1} \tag{9}
\]
Hmmm . . . looks familiar . . . aha! It’s the good old tangency condition we’ve seen so much - compare (9) to (5). Now, we can use (9) to solve \(x_1\) in terms of \(x_2\) and plug that into (8), and from there on, it’s the same process we did before.

### 3.2 A general result

In general, if a utility function is of the form
\[
u(x_1, x_2) = cx_1^\beta x_2^\gamma
\]
the demand functions will be
\[
x_1(p_1, p_2, m) = \frac{\beta}{\beta + \gamma} \frac{m}{p_1} \quad \text{and} \quad x_2(p_1, p_2, m) = \frac{\gamma}{\beta + \gamma} \frac{m}{p_2}
\]

### 4 Quasilinear

Quasilinear preferences are a bit harder to deal with, because we have to consider two possibilities - interior solutions and border solutions. We also have the additional complication that the demand of one of the goods is independent of wealth at almost all points. Remember quasilinear functions have the general form
\[
u(x_1, x_2) = v(x_1) + x_2
\]
where \(x_2\) enters the utility function directly and has a marginal utility of 1 and \(x_1\) enters the utility function via some other function \(v\).

Both examples here will use the tangency condition - check your class notes (from 11/8) and the appendix of chapter 6 of your textbook for a derivation using the Lagrange method.

**Problem**: Let there be a utility function of the form \(u(x_1, x_2) = 3\sqrt{x_1} + x_2\). Find the demand functions.
**Solution** : Since we have convexity, we can use the tangency condition.

\[ \frac{-p_1}{p_2} = -\frac{3}{2} \cdot \frac{x_1^{-0.5}}{1} = -\frac{3}{2\sqrt{x_1}} \]

\[ x_1(p_1, p_2, m) = \left( \frac{3}{2} \cdot \frac{p_2}{p_1} \right)^2 \]  

(10)

Note that we’ve already gotten a demand function for \( x_1 \), and income doesn’t enter into the function at all. Now to solve for the demand of \( x_2 \), we plug (10) into the budget constraint and solve for \( x_2 \).

\[ m = p_1 \left( \frac{3}{2} \cdot \frac{p_2}{p_1} \right)^2 + p_2 x_2 \]

\[ m = \frac{9}{4} \cdot \frac{p_2^2}{p_1} + p_2 x_2 \]

\[ x_2(p_1, p_2, m) = \frac{m}{p_2} - \frac{9}{4} \cdot \frac{p_2}{p_1} \]  

(11)

So there we have our demand functions - equations (11) and (10).

**Problem** : Say we have the utility function \( u(x_1, x_2) = 20x_1 - 4x_1^2 + x_2 \), an income of 200, and prices \( p_1 = 4 \) and \( p_2 = 1 \). What is our demand for good 1? What about good 2? For what values of \( m \) would we consume only good 1?

**Solution** : Following a similar process to that done above

\[ \frac{-p_1}{p_2} = -\frac{20 - 8x_1}{1} \]  

(12)

\[ x_1(p_1, p_2, m) = \frac{5}{2} - \frac{p_1}{8p_2} \]  

(13)

Like the last problem, we now have a demand function where wealth is nowhere to be found. Does that mean that demand for \( x_1 \) is never affected by wealth? Not entirely, and we’ll see why in a second. Plug (13) into the budget constraint and solve for \( x_2 \).

\[ m = p_1 \left( \frac{5}{2} - \frac{p_1}{8p_2} \right) + p_2 x_2 \]

\[ x_2(p_1, p_2, m) = \frac{m}{p_2} - \frac{5p_1}{2p_2} + \frac{1}{8} \left( \frac{p_1}{p_2} \right)^2 \]  

(14)
Plugging in our known values, we get

\[
x_1(p_1, p_2, m) = \frac{5}{2} - \frac{4}{8} = 2
\]

\[
x_2(p_1, p_2, m) = 200 - \frac{20}{2} + \frac{1}{8} \left( \frac{16}{1} \right) = 188
\]

Now what about this “when do we consume good 1” business? Remember with quasilinear preferences, we can have an interior solution or a border solution. To find out when we would consume only good 1, we need to find out for what values of \( m \) we have a border solution. That happens when (under the current prices) the demand for \( x_2 \) comes out to be zero or negative.

\[
x_2(p_1, p_2, m) = \frac{m}{p_2} - \frac{5}{2} p_1 + \frac{1}{8} \left( \frac{p_1}{p_2} \right)^2 \leq 0
\]

\[
\frac{m}{p_2} \leq \frac{5}{2} p_1 + \frac{1}{8} \left( \frac{p_1}{p_2} \right)^2
\]

\[
m \leq 8
\]

So for all incomes less than 8, we’ll consume only good 1 and none of good 2, giving us a border solution. Once we have an income greater than 8, we’ll keep our demand for good 1 at 2, and spend all of our additional income on good 2. See now why we can’t say our consumption of \( x_1 \) is never affected by wealth in this problem? If our wealth is low enough that we don’t consume any \( x_2 \), then changes in wealth do change our consumption of \( x_1 \). But once \( m \) is large enough (in this case, once \( m > 8 \)) changes in wealth do not change demand for \( x_1 \).