Chapters 7 and 8 – Solow Growth Model Basics

The Solow growth model breaks the growth of economies down into basics. It starts with our production function $Y = F(K, L)$ and puts in per-worker terms.

$$\frac{Y}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right)$$

$$y = f(k)$$

(1)

where $k$ is the amount of capital per worker and $y$ is the amount of output per worker. The slope of this function measures the change in output per worker due to a one unit increase in capital per worker which, as we saw from chapter 3, is equal to the $MPK$. Thus the slope of (1) is $f'(k) = MPK$. Due to the decreasing marginal productivity of capital, this is decreasing in $y$, making $f(k)$ a concave function.

Individuals consume whatever they do not save, where $s$ is the savings rate, somewhere between 0 and 1

$$c = (1 - s)y$$

(2)

All output is either allocated to consumption or investment.

$$y = c + i$$

(3)

By combining equations 2 and 3, we can show that $i = sy$.

1 The Steady State

What changes $k$? For now, we’ll look at depreciation and population growth. Population increase (denoted as a percentage by $n$) doesn’t actually affect the amount of capital ($K$) in our economy.
What it does do, however, is decrease the amount of capital per worker \( k \). Depreciation (denoted by \( \delta \)) is the rate at which capital wears out. These two factors combined are eating away at our capital per worker on a regular basis. In order to retain an unchanging level of capital per worker \( k \) over time, we have to invest enough to create new capital to offset this loss over time. Thus, to maintain a “steady state” where capital per worker is constant over time, we must have that:

\[
\Delta k = \underbrace{sf(k)}_{\text{investment in new capital}} - \underbrace{(\delta + n)k}_{\text{“loss” in capital}} = 0 \rightarrow sf(k^*) = (\delta + n)k^*
\]

where * indicates steady state values. Note how this shows that as our capital per worker \( k \) gets larger, larger amounts of investment are required to maintain \( \Delta k = 0 \). The economy will always work itself to a steady state point. If the rate of capital replenishment is greater than the loss due to depreciation and population growth \( sf(k) > (\delta + n)k \), then the capital stock will grow. If the rate of replenishment is lower than depreciation plus population growth \( sf(k) < (\delta + n)k \), then the capital stock will shrink. Only when the two are equal will there be no further adjustment to the capital stock in the economy.

There are an infinite number of possible steady states, some higher than others. Which steady state our economy is in (and therefore what output we have) depends on where the \( sf(y) \) curve meets the \( (\delta + n)k \) curve, which in turn depends on the savings rate \( s \) in the economy.

Figure 1 shows us that the higher the savings rate, the higher the capital per worker, and the higher the output per worker. Does that mean we want to save ALL our income? Of course not – if none
of that output is consumed, people starve to death. Besides, buying things is good. So what level of savings should we aim for?

2 The Golden Rule

In economics, we generally assume that the more people consume, the happier they are. So if we want people to be as happy as possible, our aim is to maximize consumption per worker \(c\). The steady state associated with that particular outcome is called the “Golden Rule” (GR) steady state. By (3), we know

\[
c^* = f(k^*) - (\delta + n)k^*
\]

While higher levels of capital mean higher levels of output, they also mean more capital is being “removed” from the economy each year. If the capital stock is below the GR level, the slope of the production function is greater than that of the capital stock curve, and an increase in capital per worker has a greater impact on \(f(k)\) than on \((\delta + n)k\) giving us an increase in consumption. The opposite will hold true when we are above the GR level. The GR steady state occurs when

\[
\frac{f'(k^*)}{MPK} = (\delta + n)
\]

Figure 2: The Golden Rule steady state - the dotted line represents the slope of the production function at the equilibrium point and the subscript “GR” indicates values are Golden Rule steady state values.
If the MPK is greater than $(\delta + n)$, we know that adding capital will increase consumption. If the MPK is less than $(\delta + n)$, we know that decreasing capital will increase consumption. Maximization of consumption occurs when (4) holds. A planner trying to maximize long-run consumption would then aim to get a savings rate that corresponded with that particular steady state level of capital.

Note that in the transition to the GR point, there will be “initial” effects and “long-run” effects. Say were below the GR. As we increase savings, there will be a temporary decrease in consumption, and then a long run increase. Why? Because an increase in savings means less consumption right away ($c = y - sy$). However, as capital accumulates, output increases, and thus so does consumption. This situation gives us a look into why it’s called the Golden Rule . . . because we sacrifice consumption now for higher consumption for the people of the future. As Mankiw puts it, the welfare of all generations is given equal weight, so sacrifice by this generation is outweighed by the gains of future generations.

3 The Addition of Effective Workers

With the Solow model thus far, there is no way to explain sustained growth in output per worker. To explain that, we have to add worker efficiency into the mix. Worker efficiency basically determines how productive workers can be at any given level of capital. It includes mechanical things like more efficient assembly lines and computer technology, and more “human” related things, like worker health and education.\footnote{The Solow growth model doesn’t make a connection between savings, investment, and technological progress. Savings affects capital stock – that’s all. Our growth in efficiency is exogenous, determined by some outside force over which the players in the Solow economy have no influence.}

We include worker efficiency $E$ by allowing it to increase the productivity of labor (for this reason, it is generally called “labor augmenting” technology). While there may be $L$ actual workers in the economy, the number of “effective workers” is $L \times E$. For example, if $E = 2$, “augmented” workers produce as much output as twice as many “non-augmented” workers. Including worker
efficiency $E$ into our model, we get

$$\frac{Y}{L*E} = F\left(\frac{K}{L*E}, \frac{L}{L*E}\right) = f(k)$$

where $k$ is now capital per effective worker, and $f(k)$ is now output per effective worker. The graph looks pretty much the same (see Figure 3), but there is now another force involved with the capital stock curve. The growth of worker efficiency over time is denoted $g$. Just as the growth in $L$ meant we needed more capital to maintain a constant capital-per-worker ratio, the growth in $E$ means we need more capital to keep a constant capital-per-effective-worker ratio.

This can get somewhat confusing . . . it looks like an increase in $g$ moves us to a lower steady state. But higher efficiency means workers are more productive, so it seems everything should be going up, not down. Remember that we have redefined our variables to be in terms of per effective worker, so it makes sense that if our efficiency increases (i.e. the number of “effective” workers in our economy increases) but our capital stock doesn’t change, we should have lower steady state per-effective-worker values. And, as we’ll see below, while output per effective worker goes down with $g$, output per worker actually goes up.

The steady state condition is now

$$sf(k) = (\delta + n + g)k$$

and the condition required for the GR point is

$$MPK = (\delta + n + g)$$

Now consider what increases in efficiency do to output per worker. Output per effective worker is

$$y = \frac{Y}{L*E}.$$ A little rearranging gives us

$$\frac{Y}{L} = y * E$$

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Figure 3: The steady state in a Solow growth model with depreciation, population growth, and growth in efficiency.

which shows us that output per worker \((Y/L)\) is growing with \(E\). As growth in worker efficiency can be maintained over time, the inclusion of this variable allows for constant long-term growth in output per worker.